The University of Chicago

SURFACES ASSOCIATED WITH A CONGRUENCE

A DISSERTATION SUBMITTED TO THE FACULTY
OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF DOCTOR
OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

By
GREENVILLE D. GORE

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CHICAGO, ILLINOIS
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INTRODUCTION

It is known that in a linear space of n dimensions, any two sub-spaces of r dimensions have in common a space of intersection of at least 2r-n dimensions. When r = 2, these sub-spaces are planes and they do not intersect, in general, unless $n \leq 4$. Hence if two planes are immersed in a space of more than four dimensions and have an intersection, they are restricted to be in a special relation to each other. This principle can be applied to surfaces by imposing restrictions on their tangent planes.

It is the purpose of this paper to study the properties of, and the configuration determined by, two surfaces in a space of n dimensions, which are in a one-to-one point correspondence such that the pairs of planes tangent to the two surfaces at corresponding points, intersect in straight lines.

It is supposed that pairs of corresponding points in this correspondence have the same curvilinear coordinates, and the pair of surfaces related by this correspondence is studied by means of a system of differential equations in the homogeneous coordinates of the generating points of the two surfaces. These equations are referred to as the differential equations of the correspondence.

The totality of correspondences considered, is sepa-

rated into three geometrically distinct types, termed hyperbolic, parabolic, and identical, respectively. In accordance with this fact, we are able, in Chapter I, to transform the differential equations of the correspondence into three canonical forms.

In Chapter II, Article 3, we show that the homogeneous coordinates of a general point on a surface S in a hyperbolic correspondence with a second surface, satisfy a special pair of third order partial differential equations; and conversely. We also show in Article 4, that pairs of corresponding points of two surfaces in a hyperbolic correspondence are joined by the lines of a congruence which has two distinct focal surfaces. This congruence of lines is conjugate to the curves of the surfaces which are parametric in the canonical differential equations of the surfaces. Every pair of surfaces transversal to this congruence is in hyperbolic correspondence.

Following a paper by B. Segre,[1] P. Tzitzéica[2] showed how to determine a congruence of lines conjugate to the parametric net of an integral surface of a system of differential equations of the type which we find represents the above surface S. By proving in Article 4 that this congruence is unique, we deduce that the totality of surfaces in hyperbolic

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1. B. Segre,"Sur l'intégration d'un certain système d'équationes différentielles,"Contes Rendus, t.184(1927),p.268.
2. P. Tzitzéica,"Sur un certain système d'équations aux dérivées partielles,"Contes Rendus, t.184(1927),p.582.

correspondence with the surface S are surfaces transversal to its conjugate congruence. This congruence initiates a sequence of Laplace which is definitely associated with the surface S.

In the latter part of Article 4, a brief account is given of a transformation by Bompiani.[1] This transformation is analogous to the classical transformation of Laplace, and is applicable to any surface which is transversal to a congruence of lines. Mention is also made of the results of Segre[2] and Tzitzéica,[3] who have shown that the sequences generated by the transformation of Bompiani are inscribed in a sequence of Laplace.

In Chapter III, we establish the existence of a much more general class of sequences, all of which are successively inscribed in a sequence of Laplace. By the study of these sequences, we are able to produce transformations analogous to the classical transformation of Laplace, for a much greater class of surfaces than the class for which such transformations have hitherto been given. In Article 6, special attention is given to the class of sequences which follow, in order of inscription, the sequences of Bompiani. By the study of these sequences, the problem of determining the integral

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1. Bompiani, "Determinazione delle superficie integrali d'un sistema di equazioni a derivate parziali, lineari, ed omogenee," Rendiconti del Reale Istituto Lombardo di Scienze e lettere, Vol. 52(1919), p. 826. (Cited hereafter as the "Lombardo Paper").
2. B. Segre, loc. cit.
3. P. Tzitzéica, loc. cit.

surfaces of certain systems of equations of the third and fourth orders is reduced to the problem of integrating a single Riccati differential equation, and a single differential equation of Laplace.

In Chapter IV, a study is made of parabolic and identical correspondences. In Article 9, the point differential equations of a general surface in a parabolic correspondence are determined, and are found to form a system of two differential equations of the third order. We show that the lines joining corresponding points of two surfaces in a parabolic correspondence are ∞^2 tangent lines to a family of asymptotic curves on a focal surface of those lines. In Article 10, it is shown that two surfaces in identical correspondence have corresponding points joined by the lines of a bundle, and conversely.

CHAPTER I

ANALYTIC FORMULATION OF THE PROBLEM

1. Differential Equations Representing the Correspondence. Consider, in a linear space of n dimensions, two surfaces in a one-to-one point correspondence such that the tangent planes to the two surfaces at any two corresponding points, intersect in a straight line. Let the two surfaces be generated by two points of which the homogeneous coordinates are $x_1, x_2, \ldots, x_{n+1}$, denoted by x; and $y_1, y_2, \ldots, y_{n+1}$, denoted by y. The coordinates x and y are to be single-valued functions of the curvilinear coordinates u,v. The parametric representation will be such that each pair of values (u,v) determines the coordinates of a pair of corresponding points x, y on the two surfaces S_x and S_y. The functions x and y are required to have as many derivatives with respect to u and v as are needed. This requirement will be made of all functions used in this paper.

It will now be shown that if the line of intersection of each pair of corresponding tangent planes passes through the point of contact y of one of these planes with its surface S_y, the surface S_x sustains a conjugate net or an asymptotic family of curves. Since the point of contact y is on the line of intersection of the two tangent planes, it is in the plane

which is tangent at x to S_x. The coordinates of any point y in this tangent plane of S_x can be expressed in the form

$$y = fx+mx_u+Sx_v,$$

in which m, $S \neq 0$, 0. If we take as a v-curve on S_y, a curve which is tangent at the point y to the line in which the corresponding tangent planes intersect, the coordinates of a second point on this line can be expressed in the form

$$y_v-By = gx+tx_u+nx_v.$$

The points y and y_v-By determine the line of intersection of the two tangent planes.

If we eliminate the y and their derivatives from the two above equations, we obtain a differential equation in x, of the form

$$(x_{uv}, x_{vv}, x_u, x_v, x) = 0.$$

It is a well known fact that a surface whose point coordinates satisfy a partial differential equation of the second order, sustains a conjugate net or an asymptotic family of curves.[1] We shall exclude this case from consideration in what follows.

When the above case is excluded, the line of intersection of a pair of corresponding tangent planes of S_x and S_y is determined by the two points in which the lines tangent at y to the u and v-curves of S_y pierce the corresponding tangent plane of S_x. The coordinates of any point distinct from the

---

1. Eisenhart, Transformations of Surfaces, p.5.

point y on the line tangent at y to the u-curve of S_y can be expressed as $y_u - Ay$, and the coordinates of a similar point on the line tangent to the v-curve of S_y can be expressed as $y_v - By$. If these two points are in the tangent plane of S_x, their coordinates satisfy the following relations:

$$(1.1)\qquad \begin{aligned} y_u - Ay &= fx + mx_u + Sx_v \qquad (m,\ S \neq 0,\ 0), \\ y_v - By &= gx + tx_u + nx_v \qquad (t,\ n \neq 0,0), \end{aligned}$$

in which A, B, f, g, m, n, S, t are scalar functions of u, and v, and the matrix

$$\left\| \begin{matrix} f & m & s \\ g & n & t \end{matrix} \right\|$$

is of rank two. The right members of (1.1) determine the coordinates of two distinct points in the tangent plane of S_x.

Any two integral surfaces S_x and S_y of equations (1.1), have the u-tangent line and the v-tangent line of S_y piercing the tangent plane of S_x in two distinct points. Since these equations are necessary and sufficient conditions that S_x and S_y be in a correspondence of the type prescribed by the problem under consideration, they are fundamental to that problem.

2. Transformations on the Differential Equations. In the point correspondence represented by equations (1.1), the pencil of lines tangent at the point x to curves on S_x, and the pencil of lines tangent at y to curves on S_y, cut out a pair of projectively related point ranges on the line of intersection of the tangent planes containing the two pencils. The

forms to which equations (1.1) can be reduced by transformations on the parameters u and v, depend on whether this projectivity of points is an identity, having every point self-corresponding; is hyperbolic, having two distinct self-corresponding points; or is parabolic, having two coincident self-corresponding points. According as this projectivity is an identity, is hyperbolic, or is parabolic; there are infinitely many families, are two families, or is one family, of curves C on S_x such that the line tangent at a general point x to a curve C, intersects the line which is tangent at the corresponding point y to the corresponding curve C'.

We shall now proceed to show, analytically, how certain of the above families of curves C and C' can be introduced as parametric curves; and to determine the forms of equations (1.1) which result from introducing these new parameters.

Let the parametric system of equations (1.1) be subjected to the transformation

$$u = u(\bar{u}, \bar{v}), \quad v = v(\bar{u}, \bar{v}),$$

in which

$$D = \begin{vmatrix} \bar{u}_u & \bar{u}_v \\ \bar{v}_u & \bar{v}_v \end{vmatrix} \neq 0.$$

In terms of these new parameters, the derivatives of x are

$$x_u = x_{\bar{u}}\bar{u}_u + x_{\bar{v}}\bar{v}_u.$$

$$x_v = x_{\bar{u}}\bar{u}_v + x_{\bar{v}}\bar{v}_v.$$

Similar expressions are obtainable for the derivatives of y.

When the above derivatives of x and y are substituted into equations (1.1), and linear combinations are taken to regain the forms of (1.1), the following new equations are obtained:

$$
\begin{aligned}
y_{\bar{u}} &= \bar{f}x+\bar{m}x_{\bar{u}}+\bar{S}x_{\bar{v}}+\bar{A}y, \\
y_{\bar{v}} &= \bar{g}x+\bar{t}x_{\bar{u}}+\bar{n}x_{\bar{v}}+\bar{B}y,
\end{aligned} \tag{2.1}
$$

in which

$$
\begin{aligned}
\bar{S} &= \tfrac{1}{D}(S\bar{v}_v^2+(m-n)\bar{v}_u\bar{v}_v-t\bar{v}_u^2), \\
\bar{t} &= -\tfrac{1}{D}(S\bar{u}_v^2+(m-n)\bar{u}_u\bar{u}_v-t\bar{u}_u^2), \\
\bar{m} &= \tfrac{1}{D}(S\bar{u}_v\bar{v}_v+m\bar{u}_u\bar{v}_v-n\bar{u}_v\bar{v}_u-t\bar{u}_u\bar{v}_u), \\
\bar{n} &= -\tfrac{1}{D}(S\bar{u}_v\bar{v}_v+m\bar{u}_v\bar{v}_u-n\bar{u}_u\bar{v}_v-t\bar{u}_u\bar{v}_u).
\end{aligned}
$$

If the coefficients of the quadratic equation

$$S\xi^2+(m-n)\xi\eta-t\eta^2 = 0 \tag{2.2}$$

are all zero, i.e. if $S = t = m-n = 0$, it follows that $\bar{S} = \bar{t} = \bar{m}-\bar{n} = 0$. Hence, referring to equations (2.1) with $\bar{S} = \bar{t} = \bar{m}-\bar{n} = 0$, it is seen that the tangent lines to the parametric curves of S_x and S_y, intersect even when the choice of parameters $\bar{u}$ and $\bar{v}$ is arbitrary. This situation corresponds to the pair of perspective pencils of tangent lines, or to the identity of points on the line of intersection of the pair of corresponding tangent planes.

In case the equation (2.2) does not vanish identically, and has two unequal roots in the ratios $\xi:\eta$, $(m-n)^2+4St \neq 0$, we can use one of these roots as the value of $\bar{u}_u/\bar{u}_v$, and the

other one for the value of $\bar{v}_u/\bar{v}_v$. These choices of values for the ratios $\bar{u}_u/\bar{u}_v$, and $\bar{v}_u/\bar{v}_v$ cause $\bar{S}$ and $\bar{t}$ to vanish. By reference to equations (2.1), with $\bar{S} = \bar{t} = 0$, it is seen that the $\bar{u}$-tangent lines of S_x intersect the $\bar{u}$-tangent lines of S_y; and similarly, the $\bar{v}$-tangent lines of S_x intersect the $\bar{v}$-tangent lines of S_y. This case corresponds to the hyperbolic projectivity of points on the line of intersection of the pair of corresponding tangent planes of S_x and S_y.

If, however, the equation (2.2) has equal roots so that $(m-n)^2+4St = 0$, we can choose the common value of these roots as the value of $\bar{u}_u/\bar{u}_v$, and any other value for the ratio $\bar{v}_u/\bar{v}_v$. The common value of these roots is $(m-n)/2t$, and when it is substituted into the expressions for $\bar{m}$ and $\bar{n}$, it can be seen that $\bar{m}-\bar{n} = 0$. The same substitution into $\bar{t}$ gives $\bar{t} = 0$. But $\bar{S}$ remains different from zero under this substitution. Reference to equations (2.1), with $\bar{t} = 0$, $\bar{m} = \bar{n}$, and $\bar{S} \neq 0$ shows that the $\bar{v}$-tangent lines of S_x and S_y intersect; but that the u-tangent lines of these surfaces do not intersect.

The above three cases have analytic and geometric properties sufficiently distinct to merit separate consideration. In what follows, we shall refer to these cases in accordance with the following

Definition: A one-to-one point correspondence between two surfaces, in which the tangent planes to the two surfaces at corresponding points intersect in straight lines, will be

called an identical correspondence, a hyperbolic correspondence, or a parabolic correspondence, according as the number of self-corresponding points on the projectively related point ranges cut out on the line of intersection of each pair of corresponding tangent planes, is infinity, two, or one.

From the results of the foregoing transformations on equations (2.1), we can associate with any hyperbolic correspondence a parametric system such that a system of differential equations representing the correspondence is given by the following:

$$(2.3)\qquad \begin{aligned} y_u &= fx+mx_u+Ay, \\ y_v &= gx+nx_v+By \qquad (m-n \neq 0). \end{aligned}$$

In a given parabolic correspondence, the parametric system can be chosen so that the correspondence is represented by a system of differential equations of the form

$$(2.4)\qquad \begin{aligned} y_u &= fx+mx_u+Sx_v+Ay, \\ y_v &= gx+mx_v+By. \qquad (S \neq 0). \end{aligned}$$

An identical correspondence can be represented by a system of differential equations of the form

$$(2.5)\qquad \begin{aligned} y_u &= fx+mx_u+Ay, \\ y_v &= gx+mx_v+By, \end{aligned}$$

where the choice of the parametric system is arbitrary.

CHAPTER II

HYPERBOLIC CORRESPONDENCE

3. The Point Differential Equations. It was shown in Chapter I that, by a suitable choice of the parametric system, a hyperbolic correspondence can be represented by a system of differential equations (2.3), which we rewrite for convenience as

$$(3.1)\qquad \begin{aligned} y_u &= fx+mx_u+Ay, \\ y_v &= gx+nx_v+By \end{aligned} \qquad (m \neq n).$$

From these equations, we shall determine the differential equations which must be satisfied by the x alone, in order that S_x admit a hyperbolic correspondence with a surface S_y.

If we take the derivative with respect to v of the first of equations (3.1), and the derivative with respect to u of the second of equations (3.1), we have, including equations (3.1), a system of four linear, nonhomogeneous equations in the variables y, y_u, y_v, y_{uv}. By solving these, we obtain, on setting $A_v-B_u = \Delta$,

$$(3.2)\qquad \begin{aligned} \Delta y &= (g_u-f_v+Bf-Ag)x+(g-m_v+Bm)x_u+(n_u-f-An)x_v+(n-m)x_{uv}, \\ \Delta y_u &= (f(A_v-B_u+AB)+A(g_u-f_v-Ag))x+(m(A_v-B_u+AB)+A(g-m_v))x_u \\ &\quad +A(n_u-f-An)x_v+A(n-m)x_{uv}, \\ \Delta y_v &= (g(A_v-B_u-AB)+B(Bf+g_u-f_v))x+B(Bm+g-m_v)x_u \\ &\quad +(n(A_v-B_u-AB)+B(n_u-f))x_v+B(n-m)x_{uv}. \end{aligned}$$

From the first of equations (3.2), it is evident that

if $\Delta \neq 0$, the surface S_x has a conjugate net parametric. We are primarily interested in the case for which $\Delta \neq 0$.

When equations (3.2) are subjected to the integrability conditions $\partial y/\partial u = y_u$ and $\partial y/\partial v = y_v$, it follows that the coordinates x satisfy a pair of third order partial differential equations of the form

$$\begin{aligned} &\frac{n-m}{\Delta}x_{uvv}+\left(\frac{\partial}{\partial v}\left(\frac{n-m}{\Delta}\right)-B\frac{n-m}{\Delta}+\frac{g+Bm-m_v}{\Delta}\right)x_{uv} \\ &\quad +\frac{n_u-f-An}{\Delta}x_{vv}+(x_u,x_v,x) = 0, \\ &\frac{n-m}{\Delta}x_{uuv}+\frac{g+Bm-m_v}{\Delta}x_{uu}+\left(\frac{\partial}{\partial u}\left(\frac{n-m}{\Delta}\right)-A\frac{n-m}{\Delta}+\frac{n_u-f-An}{\Delta}\right)x_{uv} \\ &\quad +(x_u,x_v,x) = 0. \end{aligned} \tag{3.3}$$

The integrability conditions on the coefficients of equations (3.3), as will be given later in equations (3.5), are all satisfied.

It now remains to show under what conditions an integral surface of a system of differential equations (3.3) will admit a hyperbolic correspondence with a second surface. A system of equations of type (3.3) can be written, in terms of general coefficients, as follows:

$$\begin{aligned} x_{uvv} &= ax_{uu}+2bx_{uv}+cx_{vv}+dx_u+ex_v+lx, \\ x_{uuv} &= a'x_{uu}+2b'x_{uv}+c'x_{vv}+d'x_u+e'x_v+l'x. \end{aligned} \tag{3.4}$$

The integrability conditions on the coefficients of equations (3.4), as given by Bompiani,[1] are the following:

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1. Bompiani, "The Lombardo Paper," p.826.

$$a = c' = 0,$$

$$c^2+c_u-2b'c-e' = 0,$$

$$a'^2+a'_v-2a'b-d = 0,$$

$$2bc+2b_u+e-2a'b'-2b'_v-d' = 0, \tag{3.5}$$

$$2bd'+cd+d_u+l-a'd'-2b'd-d'_v = 0,$$

$$2be'+ce+e_u-l'-a'e'-2b'e-e'_v = 0,$$

$$2bl'+cl+l_u-a'l'-2b'l-l'_v = 0.$$

We list the following functions of the coefficients of equations (3.4), which are invariant under the group of transformations $x = \lambda(u,v)\bar{x}$, and to which we shall refer in terms of the symbols here defined:

$$H \equiv -2b_u+2bc+e,$$

$$K \equiv -2c_v+2bc+e,$$

$$\textcircled{H}_1 \equiv 1+\partial/\partial v(c_v+a'c-2bc-e)-(2b-a')(c_v+a'c-2bc-e),$$

$$\textcircled{H} \equiv c_v-2bc-e-a'_u+2a'b'+d', \tag{3.6}$$

$$H' \equiv -2b'_v+2a'b'+d',$$

$$K' \equiv -2a'_u+2a'b'+d',$$

$$\textcircled{H}'_1 \equiv 1'+\partial/\partial u(a'_u+a'c-2a'b'-d')$$
$$-(2b'-c)(a'_u+a'c-2a'b'-d').$$

Equations (3.2) show that any surface S_y which is in a hyperbolic correspondence with S_x, is generated by a point whose coordinates are expressible as a linear combination of x, x_u, x_v, x_{uv}, in which the coefficient of x_{uv} is not zero. Let the coordinates y be expressed, in terms of undetermined coefficients, as

$$y = x_{uv}-px_u-qx_v+rx.$$

By computing the derivatives of y, and reducing the derivatives of x by means of equations (3.4), we obtain the relations

$$\begin{aligned} y_u-Ay &= (a'-p)x_{uu}+(2b'-A-q)x_{uv}+(e'-q_u+Aq)x_v \\ &\quad +(d'+p_u+r-Ap)x_u+(l'+r_u-Ar)x, \\ y_v-By &= (c-q)x_{vv}+(2b-B-p)x_{uv}+(d-p_v+Bp)x_u \\ &\quad +(e-q_v+r-Bq)x_v+(l+r_v-Br)x. \end{aligned} \tag{3.7}$$

In order that these equations reduce to a system of type (3.1), which represents a hyperbolic correspondence between S_x and S_y, it is necessary that the coefficients of x_{uu}, x_{uv}, and x_{vv} vanish from both of equations (3.7). This condition gives

$$a' = p, \qquad A = 2b'-c,$$

$$c = q, \qquad B = 2b-a'.$$

By substituting these values of p and q into the above expression for y, we obtain

$$y = x_{uv}-a'x_u-cx_v+rx. \tag{3.8}$$

From equation (3.8), we observe that <u>the generating points of all surfaces S_y which are in hyperbolic correspondence with S_x, lie on a straight line determined by the points x and $x_{uv}-a'x_u-cx_v$</u>. When the above determined values of A, B, p, q are used, equations (3.7) become

$$\begin{aligned} y_u-(2b'-c)y &= (l'+r_u+r(2b'-c))x+(r-a'_u+2a'b'+d'-a'c)x_u \\ &\quad +(e'_v-c_u-c^2+2b'c)x_v, \\ y_v-(2b-a')y &= (l+r_v-r(2b-a'))x+(d-a'_v+2a'b-a'^2)x_u \\ &\quad +(r-c_v+e+2bc-a'c)x_v. \end{aligned} \tag{3.9}$$

The coefficient of x_v in the first of (3.9) is zero, by the first of integrability conditions (3.5); and the coefficient of x_u in the second of (3.9) is zero, by the third of equations (3.5).

Equations (3.9) are of the same form as equations (3.1) if and only if we append the condition corresponding to $m-n \neq 0$, of (3.1). In terms of the coefficients of (3.4), this condition is that the invariant

$$\Theta = c_v - 2bc - e - a'_u + 2a'b' + d'$$

be not zero.

These results are summarized by

Theorem 1. If a surface S_x does not sustain a conjugate net, or an asymptotic family of curves, a necessary and sufficient condition that it admit a hyperbolic correspondence with a second surface is that the coordinates x of its generating point satisfy a system of differential equations (3.4) for which the invariant Θ is different from zero.

4. The Congruence of Lines Joining Corresponding Points. In a space of n dimensions, a congruence of lines is defined[1] as a two parameter family of lines such that through each line of the family pass two developable surfaces which are formed by lines of the family. In these terms, we shall establish

Theorem 2. If two surfaces are in a hyperbolic correspondence, the lines joining corresponding points of the two

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1. Eisenhart, loc. cit., p.11.

surfaces form a congruence with distinct focal surfaces.

Let S_x and S_y be in a hyperbolic correspondence which is represented by equations (3.1). Consider the surfaces generated by the points z and z', which have coordinates defined as

$$(4.1)\qquad \begin{aligned} z &= y-mx, \\ z' &= y-nx. \end{aligned}$$

By differentiating z and z', and using the resulting forms to eliminate y from equations (3.1), we obtain the following relations:

$$(4.2)\qquad \begin{aligned} z_u-Az &= (-m_u+Am+f)x, \\ z'_v-Bz' &= (-n_v+Bn+g)x. \end{aligned}$$

Equations (4.2) show that the line xy is tangent to the u-curve of S_z, and to the v-curve of S_z'. Hence this line is a generator of two developable surfaces. It is then, by definition, a generator of a congruence.

The converse of Theorem 2 is expressed by

Theorem 3. If two surfaces are in a one-to-one point correspondence such that corresponding points are joined by the lines of a congruence having distinct focal surfaces, the surfaces are in a hyperbolic correspondence.

Let the focal surfaces of the congruence be denoted by S_z and S_z'. If corresponding points of two surfaces S_x and S_y are joined by the lines of this congruence, the coordinates x, y, z, and z' satisfy a relation of the same form as equations (4.1), in which $m \neq n$. If the focal curves of S_z and S_z'

be chosen as the parametric u and v-curves respectively, the coordinates z, z', and x satisfy relations of form (4.2), for suitably chosen values of A, B, f, g. By eliminating z and z' from equations (4.1) and (4.2), we obtain a system of equations of the form of (3.1), which represent a hyperbolic correspondence between S_x and S_y.

Corollary. Every two transversal surfaces of a congruence, both of which are distinct from the focal surfaces of the congruence, are in a hyperbolic correspondence.

A congruence of lines is conjugate[1] to a system of curves on a surface if the generating line of the congruence does not lie in the plane which is tangent to the surface at the point where the generator intersects the surface, and if the developable surfaces of the congruence intersect the surface in the curves of the system.

A method was given by B. Segre[2] and P. Tzitzeica,[3] for determining a congruence of lines conjugate to the parametric net of an integral surface of a system of equations (3.4). Since, by equation (3.8), it was observed that the generating points of all surfaces in hyperbolic correspondence with an integral surface S_x of these equations, lie on a straight line, we have, by Theorems 2 and 3,

Theorem 4. To the parametric net of an integral

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1. Eisenhart, loc. cit., p. 12.
2. B. Segre, loc. cit.
3. P. Tzitzéica, loc. cit.

surface of a system of differential equations (3.4) for which the invariant Θ is not zero, there is a unique conjugate congruence of lines.

We now proceed to determine the coordinates of the focal surfaces of the congruence conjugate to the parametric net of S_x, and to determine analytic expressions for the relations of these surfaces to S_x.

In Article 3, we showed that all surfaces in hyperbolic correspondence with S_x are generated by points y, where the coordinates y are defined in terms of a parameter r, by equation (3.8). We also showed that these coordinates satisfy equations (3.9), which reduce to the forms

$$(4.3)\qquad \begin{aligned} y_u-(2b'-c)y &= (l'+r_u+r(2b'-c))x+(r-a'_u+2a'b'+d'-a'c)x_u,\\ y_v-(2b-a')y &= (l+r_v-r(2b-a'))x+(r-c_v+e+2bc-a'c)x_v. \end{aligned}$$

If we denote by η, the special values of the coordinates y obtained by setting $r = c_v+a'c-2bc-e$ in equation (3.8), we have

$$(4.4)\qquad \eta = x_{uv}-a'x_u-cx_v+(c_v+a'c-2bc-e)x.$$

By substituting these values of r and y into equations (4.3), we obtain the relations

$$(4.5)\qquad \begin{aligned} \eta_u-(2b'-c)\eta &= \Theta(x_u-cx),\\ \eta_v-(2b-a')\eta &= \Theta_1 x. \end{aligned}$$

The coefficient of x in the first of equations (4.5), as first determined, is $l'+\partial/\partial_u(c_v+a'c-2bc-e)-(2b'-c)(c_v+a'c-2bc-e)$. In order to factor this into the form $-c\Theta$, it is necessary to make use of the second, fourth, and sixth of integrability

conditions (3.5).

Equations (4.5) show that the v-curve of S_η is tangent to the line $x\eta$ if $\Theta_1 \neq 0$. S_η is then a v-focal surface of the congruence of lines $x\eta$, which is conjugate to the parametric net of S_x. If Θ_1 vanishes, the surface S_η degenerates to a curve on which

$$\eta_v-(2b-a')\eta = 0.$$

By using $r = a'_u+a'c-2a'b'-d'$ in equations (3.8), and by making substitutions and reductions similar to the above, we can determine the coordinates η' of the u-focal surface of the congruence of lines $x\eta$. These are

(4.6) $\eta' = x_{uv}-a'x_u-cx_v+(a'_u+a'c-2a'b'-d')x.$

These coordinates satisfy the relations

(4.7)
$$\eta'_u-(2b'-c)\eta' = \Theta'_1 x,$$
$$\eta'_v-(2b-a')\eta' = \Theta(x_v-a'x).$$

It is convenient at this point to call attention to a transformation, given by Bompiani,[1] for the integral surfaces of a system of equations (3.4).

From the first of equations (4.5), it is evident that the u-tangent line of S_η intersects the u-tangent line of S_x in a point x_{-1}, whose coordinates are proportional to

(4.8) $x_{-1} = x_u-cx.$

From the second of equations (4.7) it is apparent that the v-tangent line of $S_{\eta'}$ intersects the v-tangent line of S_x in a

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1. Bompiani, "The Lombardo Paper," p.829.

point x_1, whose coordinates are proportional to

(4.9) $x_1 = x_v - a'x.$

Since the surface $S_{x_{-1}}$ generated by the point x_{-1}, is transversal to the congruence of u-tangent lines of S_y, the surface $S_{x_{-1}}$ is in hyperbolic correspondence with other surfaces, by Theorem 3; and the coordinates x_{-1} satisfy a system of equations (3.4), by Theorem 1. Hence $S_{x_{-1}}$ is a transform of S_x, and has the same essential geometric properties as S_x. For reasons similar to those given for $S_{x_{-1}}$, the surface S_{x_1} is also a transform of S_x.

If a transformation of this kind is applied to each of $S_{x_{-1}}$ and S_{x_1}, and then to the transforms of these, a sequence of surfaces, similar to a classical sequence of Laplace, is generated. We shall refer to it as a sequence of Bompiani.

It was first noted by P. Tzitzéica[1] that this sequence of surfaces

(4.91) ..., $S_{x_{-1}}$, S_x, S_{x_1}, ...,

generated by transformations (4.8) and (4.9), is inscribed in the sequence of Laplace which is initiated by the congruence of lines conjugate to the parametric net of S_x. The developable surfaces of the congruences of lines connecting the surfaces of the sequence of Laplace, cut out the parametric nets on the surfaces of the sequence of Bompiani. These facts are made analytically evident by equations (4.5) and (4.7).

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1. P. Tzitzéica, loc. cit.

If any surface, say S_x, of the sequence of surfaces (4.91) has a conjugate net for a parametric system, the Bompiani transformations of this surface are identical with the Laplace transformations. To show this, let the coordinates x satisfy an equation

$$x_{uv} = a'x_u + cx_v + \lambda x.$$

By taking u and v-derivatives of this equation, we obtain the relations

$$x_{uvv} = cx_{vv} + a'x_{uv} + (x_u, x_v, x),$$
$$x_{uuv} = a'x_{uu} + cx_{uv} + (x_u, x_v, x).$$

The Bompiani transformations (4.8) and (4.9) for the two third order equations in x are exactly the two Laplace transformations for the second order equation in x.

CHAPTER III

SEQUENCES OF AN EXTENDED CLASS

5. The Existence of an Extended Class of Sequences.

After seeing in Article 5 that a sequence of Bompiani is inscribed in a sequence of Laplace, it is natural to ask if there are any sequences of similar characters inscribed in a sequence of Bompiani. A brief consideration will show that there is in general a very large class of sequences successively inscribed in a given sequence of Laplace, and that of these, the sequences of Bompiani are first in order of simplicity and in order of inscription.

To show the existence of the above mentioned class of sequences, let the surfaces

(5.10) $\cdots, S_{x_i}, S_{x_{i-1}}, \cdots, S_{x_1}, S_x, S_{x_{-1}}, \cdots, S_{x_{-j}}, \cdots,$

be the surfaces of any sequence in which the surfaces have similar geometric properties, and in which each surface is generated by a point on the parametric tangent line of the preceding surface. The coordinates of the generating points of these surfaces will be related by equations of the forms

(5.11)
$$x_{r+1} = x_{rv} - a'_r x_r,$$
$$x_{-r-1} = x_{-r}u - c_{-r}x_{-r} \qquad (r > 0),$$

and

$$x_1 = x_v - a'x,$$
$$x_{-1} = x_u - cx.$$

Let S_z be any surface which is transversal to the lines xx_{-1}. The coordinates z can be expressed as

(5.12) $z = x_u - \lambda x$ $\qquad (\lambda = c).$

By differentiation one obtains

$$z_u = x_{uu} - \lambda x_u - \lambda_u x,$$

$$x_{-1u} = x_{uu} - c x_u - c_u x.$$

The four functions z, z_u, x_{-1}, x_{-1u} are expressed linearly in terms of the three functions x_{uu}, x_u, x. By eliminating the latter, we obtain a relation of the form

(5.13) $$z_u - \left(\frac{c^2 - c\lambda + c_u - \lambda_u}{c - \lambda}\right) z = x_{-1u} - \left(\frac{c\lambda - \lambda^2 + c_u - \lambda_u}{c - \lambda}\right) x_{-1}.$$

The left member of equation (5.13) defines the coordinates of a point on the u-tangent line of S_z, while the right member of (5.13) defines the coordinates of a point on the u-tangent line of $S_{x_{-1}}$. The surface $S_{z_{-1}}$ generated by this point, is transversal to the u-tangent lines of $S_{x_{-1}}$. The relation of $S_{z_{-1}}$ to $S_{x_{-1}}$ is similar to the relation between S_z and S_x. Hence $S_{z_{-1}}$ is a geometric transform of S_z. In a similar manner, the point in which the u-tangent line of $S_{z_{-1}}$ intersects the u-tangent line of $S_{z_{-2}}$ generates a transform of $S_{z_{-1}}$, and so on.

The relations between the coordinates of the generating points of the surfaces S_z, $S_{z_{-1}}$, ..., are of the form

(5.14) $$z_{-1} = z_u - \gamma_0 z \qquad (\gamma_0 = (c^2 - c\lambda + c_u - \lambda_u)/(c - \lambda)),$$

$$z_{-2} = z_{-1} - \gamma_{-1} z_{-1},$$

$$\cdots\cdots\cdots\cdots\cdots$$

By a repetition of the argument concerning the sequence of surfaces S_x, S_{x-1}, ..., it follows that there are sequences of surfaces inscribed in the sequence of surfaces S_z, $S_{z_{-1}}$, ...,

and so on for each inscribed sequence.

6. Inscribed Sequences of the Third Order. We shall refer to a general sequence of surfaces inscribed in a sequence of Bompiani as an inscribed sequence of the third order if it is not a sequence of Bompiani.

Let it be assumed that the sequence of surfaces (5.10) is a sequence of Bompiani. We shall then determine the point differential equations of the surface S_z which is transversal to the lines xx_{-1}; and, from the differential equations obtained, study the sequence of surfaces $z, z_{-1}, \ldots,$.

The coordinates of the generating point of the surface S_x satisfy a pair of equations (3.4), which we rewrite here for convenience as

$$(6.10)\qquad \begin{aligned} x_{uvv} &= 2bx_{uv}+cx_{vv}+dx_u+ex_v+lx, \\ x_{uuv} &= a'x_{uu}+2b'x_{uv}+d'x_u+e'x_v+l'x, \end{aligned}$$

subject to integrability conditions (3.5), and to the condition that the invariant Θ, in equations (3.6), is not zero.

From the coordinates z, as defined by equation (5.12), we list the following derivatives, the right members of which have been reduced by equations (6.10):

$$
\begin{aligned}
&z = x_u - \lambda x \qquad (\lambda \neq c),\\
&z_u = x_{uu} - \lambda x_u - \lambda_u x,\\
&z_v = x_{uv} - \lambda x_v - \lambda_v x,\\
(6.11)\qquad &z_{uv} = a' x_{uu} + (2b' - \lambda) x_{uv} + (d' - \lambda_v) x_u + (e' - \lambda_u) x_v + (1' - \lambda_{uv}) x,\\
&z_{vv} = (c - \lambda) x_{vv} + 2b x_{uv} + d x_u + (e - 2\lambda_v) x_v + (1 - \lambda_{vv}) x,\\
&z_{uu} = x_{uuu} - \lambda x_{uu} - 2\lambda_u x_u - \lambda_{uu} x,\\
&z_{uvv} = (x_{uu}, x_{uv}, x_{vv}, x_u, x_v, x),\\
&z_{uuv} = a' x_{uuu} + (x_{uu}, x_{uv}, x_u, x_v, x).
\end{aligned}
$$

The left members of the eight equations (6.11) are expressed linearly in terms of the seven functions x, x_u, x_v, x_{uu}, x_{uv}, x_{vv}, x_{uuu}. Hence the functions on the left satisfy a linear relation. That relation is found to be of the form

$$(6.12)\qquad \Delta_1 (z_{uuv} - \Gamma z_{uu}) + (c - \lambda)\Delta_2 (z_{uvv} - \delta_2 z_{vv}) + (z_{uv}, z_u, z_v, z) = 0,$$

where Δ_1 is the determinant of the coefficients of the right members of z, z_u, z_v, z_{uv}, z_{vv}, z_{uvv}, in (6.11); and Δ_2 is the determinant of the coefficients of the right members of z, z_u, z_v, z_{uv}, z_{uu}, z_{uuv}, in (6.11). The coefficients Δ_1 and Δ_2 are functions of λ and its derivatives, and therefore do not vanish for all values of λ. In fact they vanish only for very specially chosen values of λ. But if Δ_1 and Δ_2 are both zero, the coordinates z satisfy a system of differential equations (6.10), and S_z belongs to a sequence of Bompiani.

Another significant list of derivatives from equations (5.12) is the following:

$$z = x_u - \lambda x \qquad (\lambda \neq c),$$

$$z_u = x_{uu} - \lambda x_u - \lambda_u x,$$

$$z_v = x_{uv} - \lambda x_v - \lambda_v x,$$

$$z_{uv} = a' x_{uu} + (2b' - \lambda) x_{uv} + (d' - \lambda_v) x_u + (e' - \lambda_u) x_v + (l' - \lambda_{uv}) x, \tag{6.13}$$

$$z_{vv} = (c - \lambda) x_{vv} + 2b x_{uv} + d x_u + (e - \lambda_v) x_v + (l - \lambda_{vv}) x,$$

$$z_{uvv} = (x_{uu}, x_{uv}, x_{vv}, x_u, x_v, x),$$

$$z_{vvv} = (c - \lambda) x_{vvv} + (x_{uv}, x_{vv}, x_u, x_v, x),$$

$$z_{uvvv} = (c(c - \lambda) + c_u - \lambda_u) x_{vvv} + (x_{uu}, x_{uv}, x_{vv}, x_u, x_v, x).$$

The above eight functions on the left are expressed in terms of the seven functions x_{vvv}, x_{uu}, x_{uv}, x_{vv}, x_u, x_v, x. Hence they satisfy a linear relation, which is found to be

$$\Delta_1 (z_{uvvv} - A z_{vvv}) + (z_{uvv}, z_{uv}, z_{vv}, z_u, z_v, z) = 0, \tag{6.14}$$

with Δ_1 as in (6.12) and $A = (c^2 - c\lambda + c_u - \lambda_u)/(c - \lambda)$.

Consider in the same manner, the list of derivatives from (5.12):

$$z = x_u - \lambda x,$$

$$z_u = x_{uu} - \lambda x_u - \lambda_u x,$$

$$z_v = x_{uv} - \lambda x_v - \lambda_v x,$$

$$z_{uv} = a' x_{uu} + (2b' - \lambda) x_{uv} + (d' - \lambda_v) x_u + (e' - \lambda_u) x_v + (l' - \lambda_{uv}) x, \tag{6.15}$$

$$z_{uu} = x_{uuu} - \lambda x_{uu} - 2\lambda_u x_u - \lambda_{uu} x,$$

$$z_{uuv} = a' x_{uuu} + (x_{uu}, x_{uv}, x_u, x_v, x),$$

$$z_{uuu} = x_{uuuu} - \lambda x_{uuu} + (x_{uu}, x_u, x),$$

$$z_{uuuv} = a' x_{uuuu} + (x_{uuu}, x_{uu}, x_{uv}, x_u, x_v, x).$$

These derivatives of z are expressed in terms of the seven functions x, x_u, x_v, x_{uv}, x_{uu}, x_{uuu}, x_{uuuu}, and therefore satisfy

a linear relation. That relation is found to be

$$\Delta_2(z_{uuuv}-a'z_{uuu})+(z_{uuv},\ z_{uu},\ z_{uv},\ z_u,\ z_v,\ z) = 0.$$

Since Δ_1 and Δ_2 are not both zero unless S_z belongs to a sequence of Bompiani, we have established

Theorem 5. If a surface S_z belongs to an inscribed sequence of the third order, the coordinates z satisfy a system of partial differential equations of the following form:

$$\begin{aligned} z_{uvvv} &= Az_{vvv}+Bz_{uvv}+Cz_{vv}+Dz_{uv}+Ez_u+Fz_v+Gz, \\ Qz_{uuv} &= A'z_{uvv}+B'z_{uu}+C'z_{uv}+D'z_{vv}+E'z_u+F'z_v+G'z, \\ z_{uuuv} &= A''z_{uuu}+B''z_{uuv}+C''z_{uu}+D''z_{uv}+E''z_u+F''z_v+G''z, \end{aligned} \tag{6.16}$$

in which Q and A' are not both zero.

It is important at this point to note that the coordinates z do not in general satisfy a differential equation of the third or fourth orders, in which the highest derivative present is a pure derivative with respect to u or a pure derivative with respect to v. From the seventh of equations (6.13), and the seventh of (6.15), it can be seen that such an equation can not exist in z unless a similar equation exists in x. This would be a specialization of S_x.

We shall now show that if $Q \neq 0$, the third equation of (6.16) is a consequence of the first two equations of that system. It will then follow by symmetry that if $A' \neq 0$, the first of equations (6.16) is a consequence of the last two. Hence it is necessary to work formally with only two of equations (6.16).

If $Q \neq 0$ in equations (6.16), the first two equations

of that system can be written in the form

$$
\begin{aligned}
z_{uvvv} &= Az_{vvv}+Bz_{uvv}+Cz_{vv}+Dz_{uv}+Ez_u+Fz_v+Gz, \\
z_{uuv} &= A'z_{uvv}+B'z_{uu}+C'z_{uv}+D'z_{vv}+E'z_u+F'z_v+G'z.
\end{aligned} \tag{6.17}
$$

By a transformation of the form $z = T\bar{z}$, in which T satisfies the relation $T_v - B'T = 0$, the new coefficient $\bar{B}$ is made to vanish. When this transformation has been made, the system of equations (6.17) requires the following conditions of integrability:

$$
\begin{aligned}
&AA'+D' = 0, \quad B' = 0, \\
&AA'B+AC'+AA'_v+A'C+D'_v+F'-A^2-A_u = 0, \\
&(A'B)_v+2C'_v+A'_{vv}+E'-AB-B_u-C = 0, \\
&E = 0, \\
&2A'_vD+A'D_v+C'_{vv}+2E'_v-AD-D_u-F-C'_vB-BE' = 0, \\
&CC'+2A'_vC+A'C_v+D'_{vv}+2F'_v+A'F+G'-AC-C_u-BD'_v-BF'-DD' = 0, \\
&E'_{vv}-BE'_v-DE'-G = 0, \\
&2A'_vF+A'F_v+C'F+F_{vv}+2G'_v+A'G-AF-F_u-BF'_v-BG'-DF' = 0, \\
&C'G+2A'_vG+A'G_v+G'_{vv}-AG-G_u-G'_vB-DG' = 0.
\end{aligned} \tag{6.18}
$$

If we compute the v-derivative of the second of equations (6.17) and eliminate the term in Z_{uvvv} from the result by means of the first of equations (6.17), we obtain a fourth order differential equation of the form

$$
\begin{aligned}
z_{uuvv} = {} & Z_{uvv}(B^2+A'_v+C')+Z_{vv}(D'_v+F'+A'C) \\
& +(z_{uv}, z_u, z_v, z).
\end{aligned} \tag{6.19}
$$

If, then, we compute the u-derivative of the second of (6.17), and eliminate the terms in Z_{uuvv} and Z_{uvv} by the above equation,

and the second of (6.17), we obtain an equation of the form

(6.20) $z_{uuuv} = (z_{uuu}, z_{uuv}, z_{uu}, z_{uv}, z_u, z_v, z)$.

It is necessary to use the first and second of inegrability conditions (6.18) to show that the coefficient of z_{vv} is zero in the last equation obtained.

In view of the above facts, we shall proceed to establish the converse of Theorem 5, as expressed in

Theorem 6: If the coordinates z of the generating point of a surface S_z satisfy an integrable system of differential equations (6.17), the surface S_z belongs to an inscribed sequence of the third order.

The proof will be established by determining the coordinates x of a surface S_x which is so related to S_z that the surface S_z is transversal to the u-tangent lines of S_x, and the coordinates x satisfy a system of equations (6.10).

From the first five and the seventh of equations (6.11), it is apparent that if S_x is related to S_z as described above, the coordinates x and the derivatives x_u, x_v, x_{uu}, x_{uv}, x_{vv}, are expressible in terms of z, z_u, z_v, z_{uv}, z_{vv}, z_{uvv}. Accordingly, let the coordinates x be expressed in terms of undetermined coefficients as

(6.21) $x = z_{uvv} + \beta z_{vv} + \gamma z_{uv} + \delta z_u + \rho z_v + \sigma z$,

where the functions β, γ, δ, ρ, σ are to be determined.

By computing the u derivative of (6.21) and reducing the derivatives on the right, we obtain

$$x_u = z_{uvv}(A'B+A'_v+C'+\beta+A'\gamma)+z_{vv}(A'C+D'_v+F'+\beta_u+D'\gamma)$$
$$+z_{uv}(AD+C'_v+E'+\gamma_u+\rho+C'\gamma)+z_{uu}\delta+z_u(E'_v+\sigma+\delta_u+F'\gamma) \tag{6.22}$$
$$+z_v(A'F+F'_v+G'+\rho_u+F'\gamma)+z(A'G+G'_v+\sigma_u+G'\gamma).$$

Since S_z is to be transversal to the u-tangent lines of S_x, the coordinates x and z must satisfy a relation of the form

$$\mu z = x_u-\lambda x. \tag{6.23}$$

If we substitute the values of x and x_u from equations (6.21) and (6.22) into (6.23), and demand that the resulting equation vanish term by term in the derivatives of z, we obtain the following equations of condition on the undetermined functions $\beta, \gamma, \delta, \rho, \sigma, \lambda, \mu$:

$$\begin{aligned}
&\beta+A'B+A'_v+C'+A'\gamma-\lambda = 0,\\
&\delta = 0,\\
&\gamma_u+\rho+C'\gamma+A'D+C'_v+E'-\lambda\gamma = 0,\\
&\beta_u+A'C+D'_v+F'+D'\gamma-\lambda\beta = 0,\\
&E'_v+\sigma+E'\gamma = 0,\\
&\rho_u+A'F+F'_v+G'+F'\gamma-\lambda\rho = 0,\\
&\sigma_u+A'G+G'_v+G'\gamma-\lambda\sigma = \mu.
\end{aligned} \tag{6.24}$$

Since it is also necessary that the coordinates x satisfy a system of equations of the form of (6.10), we proceed to determine what additional conditions are imposed on $\lambda, \mu, \beta, \gamma, \delta, \rho, \sigma$, in order to meet that necessity.

From equations (6.21) and (6.23) we list the following derivatives which have been simplified by means of equations (6.17):

$$
\begin{aligned}
&x_u-\lambda x = \mu z,\\
&(x_u-\lambda x)_v = (\mu z)_v,\\
&(x_u-\lambda x)_{vv} = (\mu z)_{vv},\\
&x = z_{uvv}+\beta z_{vv}+\gamma z_{uv}+\rho z_v+\sigma z,\\
&x_v = (A+\beta) z_{vvv}+(B+\gamma) z_{uvv}+(D+\gamma_v) z_{uv}+(C+\rho+\beta_v) z_{vv}\\
&\qquad +(F+\rho_v+\sigma) z_v+(\sigma_v+G) z,\\
&x_{vv} = (A+\beta) z_{vvvv}+(A_v+2\beta_v+C+\rho+AB+A\gamma) z_{vvv}\\
&\qquad +(D+2\gamma_v+B_v+B^2+B\gamma) z_{uvv}+(D_v+\gamma_{vv}+DB+D\gamma) z_{uv}\\
&\qquad +(C_v+2\rho_v+\beta_{vv}+F+\sigma+CB+C\gamma) z_{vv}\\
&\qquad +(F_v+\rho_{vv}+2\sigma_v+G+FB+F\gamma) z_v+(\sigma_{vv}+G_v+GB+G\gamma) z.
\end{aligned}
\tag{6.25}
$$

Preceding equation (6.21), it was noted that x_v and x_{vv} are expressible in terms of z, z_u, z_v, z_{uv}, z_{vv}, z_{uvv}. From this fact, it is necessary that the coefficients of z_{vvv} and z_{vvvv} vanish in the last two of equations (6.25). This gives two additional equations of condition on the functions ρ, β, γ:

$$
\begin{aligned}
&A+\beta = 0,\\
&\rho+A_v+2\beta_v+C+AB+A\gamma = 0.
\end{aligned}
\tag{6.26}
$$

When these conditions are imposed on (6.25), the six functions of x in the left members are expressed in terms of the five functions z, z_v, z_{uv}, z_{vv}, z_{uvv}. Hence the functions in the left members satisfy a linear relation. That relation is an equation of the form of the first of equations (6.10) in x.

To show that under these conditions the coordinates x also satisfy an equation of the form of the second of (6.10), we list the following functions and their derivatives:

$$x = z_{uvv}+\beta z_{vv}+\gamma z_{uv}+\rho z_v+\delta z,$$

$$x_v = (B+\gamma)z_{uvv}+(D+\gamma_v)z_{uv}+(C+\rho+\beta_v)z_{vv}$$
$$+(F+\rho_v+\delta)z_v+(\delta_v+G)z,$$

(6.27) $$x_u-\lambda x = \mu z,$$

$$(x_u-\lambda x)_u = \mu z_u+\mu_u z,$$

$$(x_u-\lambda x)_v = \mu z_v+\mu_v z,$$

$$(x_u-\lambda x)_{uv} = \mu z_{uv}+\mu_v z_u+\mu_u z_v+\mu_{uv} z.$$

The determinant of the coefficients of z, z_u, z_v, z_{uv}, z_{vv}, z_{uvv}, in the right members of (6.27), has the value

$$\mu^4(\rho+\beta_v-B\beta-\beta\gamma+C),$$

which is zero by the combination of equations (6.26). Hence the left members of (6.27) are dependent. This is equivalent to an equation of the form of the second of (6.10).

We have shown that equations (6.24) and (6.26) are the necessary and sufficient conditions on the functions β, γ, δ, ρ, σ, λ, μ, in order that S_x have the properties and relations required by Theorem 6. It remains to show that there exists a set of functions which satisfy these conditions. We shall prove this existence and give a method for determining the functions.

Take γ as any solution of the Riccati differential equation

(6.28) $$\gamma_u = A'\gamma^2+(A'_v+A'B)\gamma+AB+C-A_v-C'_v-E'-A'D.$$

In terms of γ and the coefficients of equations (6.17), define the following functions:

$$
\begin{aligned}
\beta &= -A, \\
\delta &= 0, \\
\rho &= -(\beta_v + C + AB + A\gamma), \\
\sigma &= -E'\gamma - E'_v, \\
\lambda &= \beta + A'B + A'_v + C' + A'\gamma, \\
\mu &= \sigma_u + A'B + G'_v + G'\gamma - \lambda\sigma.
\end{aligned} \tag{6.29}
$$

These functions automatically satisfy the first, second, fifth, and seventh of equations (6.24).

By substituting the above values of β and λ into the fourth of (6.24), the left member is reduced to

$$(AA' + D')\gamma - A_u - A^2 + AA'B' + AA'_v + AC' + A'C + D'_v + F'.$$

The coefficient of γ is zero by the first of integrability conditions (6.18), while the terms not containing γ are zero by the second of equations (6.18). Hence the fourth of (6.24) is satisfied.

By substituting the values of ρ, β, λ, from the equations (6.29) into the third of equations (6.24), the left member of that equation becomes

$$\gamma_u - A'\gamma^2 - (A'B + A'_v)\gamma + A_v - C - AB + A'D + C'_v + E',$$

which is zero by the definition of γ in (6.28).

When the values ρ, λ, β, from (6.29), are substituted into the sixth of equations (6.24), the left member becomes

$$
\begin{aligned}
&AA'\gamma^2 + (A'C + 2AA'B - AA'_v - A^2 + AA'_y + AC' + A_u + F')\gamma \\
&+ A_{uv} - C_u - A_u B - AB_u + A'F + F'_v + G' \\
&- (-A - A'B + A'_v + C')(A_v - C - AB) - A\gamma_u.
\end{aligned}
$$

If we eliminate γ_u from the above expression by means of equation (6.25), the result is a linear expression in γ. The coefficient of γ in this linear function is

$$AA'B+A'C-A'A_v-A^2+AC'-A_u+F'.$$

By differentiating the first of equations (6.18) we obtain $D'_v+AA'_v+A'A_v = 0$. Add this to the above coefficient of γ, and the result is indicated to be zero by the second of equations (6.18). The terms of degree zero in the above mentioned function of γ are

$$\begin{aligned}(6.30)\quad &-2A^2B-AC+2AA_v+AC'_v+AE'+AA'D+A_{uv}-C_u-A_uB-AB_u+A'F+F'_v+G' \\ &-AC-A'A_vB+A'BC+A'AB^2-A'_vA_v+A'_vC+A'_vAB-A_vC'+CC'+ABC'.\end{aligned}$$

To show that this expression is zero, take the derivative with respect to v of the second of (6.18); take -A times the third of (6.18); take -1 times the sixth of (6.18); and take -B times the second of (6.18). Add these four quantities to the above array of terms and the result is zero.

Conditions (6.26) are obviously satisfied by the functions defined by (6.29). Since there exist solutions γ of the Riccati differential equation (6.28), the foregoing investigation has established the existence of the surface S_x, as required by Theorem 6; and has given a method for determining the coordinates of the generating point of S_x. This method is stated in

Theorem 7. If S_z is an integral surface of a system of equations (6.17), define in terms of the coordinates z

a set of coordinates x by equation (6.21), in which the function γ is a solution of the Riccati differential equation (6.28), and the functions β, δ, ρ, σ are defined by equations (6.29). Then S_x is an initial surface of a sequence of Bompiani, while S_z is an initial surface of a sequence of the third order, and the u-branch of that sequence is inscribed in the sequence of Bompiani which is initiated by S_x.

Since the above method does not give a unique value for γ, it does not give a unique determination of the surface S_x which initiates the sequence of Bompiani in which is inscribed the u-branch of the sequence of surfaces S_z, $S_{z_{-1}}$, ...,. If we replace in (6.21) the values of β, δ, ρ, σ, as defined by equations (6.29), we obtain the following values of x:

$$x = z_{uvv} - Az_{vv} + (A_v - C - AB)z_v - E'_v z + \gamma(z_{uv} - Az_v - E'z). \tag{6.31}$$

This equation shows that the generating points x of all surfaces S_x lie on a straight line which is determined by the two points having coordinates

$$\begin{aligned} \xi &= z_{uvv} - Az_{vv} + (A_v - C - AB)z_v - E'_v z, \\ \zeta &= z_{uv} - Az_v - E'z. \end{aligned} \tag{6.32}$$

It can be shown that the line $\xi\zeta$ is the generating line of the congruence which is conjugate to the parametric net of S_x. To show this, we employ equations (6.17), (6.18), and the function (6.30) to express the derivatives of ξ and ζ as follows:

$$\xi_v := \theta\xi + \phi\zeta + \psi z_v,$$
$$\zeta_v = \theta'\xi + \phi'\zeta + \psi' z_v$$

The generating point of a focal surface of the congruence of lines $\xi\zeta$ would have coordinates expressible as

(6.33) $\eta = R'\xi + R\zeta.$

Then

$$\eta_v = R'\xi_v + R\zeta_v + R'_v\xi + R_v\zeta,$$
$$= P\xi + Q\zeta + (R'\psi + R\psi')z_v.$$

If R' and R be chosen proportional to ψ' and $-\psi$ respectively, the term in z_v vanishes and the point η is clearly a v-focal point on the line $\xi\zeta$.

In a similar manner, the u-derivatives of ξ and ζ can be expressed as

$$\xi_u = \tau\xi + \omega\zeta + \pi z,$$
$$\zeta_u = \tau'\xi + \omega'\zeta + \pi' z.$$

The coordinates η' of the u-focal point on the generating line $\xi\zeta$ can be expressed as

$$\eta' = S'\xi + S\zeta.$$

Then

$$\eta'_u = P'\xi + Q'\zeta + (S'\pi + S\pi')z.$$

By choosing S' and S proportional to π' and $-\pi$ respectively, the coefficient of z in the expression for η'_u vanishes, showing that for this choice of S and S', $S\eta'$ is a u-focal surface of the congruence of lines $\xi\zeta$. The one parameter family of surfaces S_x whose u-tangent lines pass through the point z are conjugate to this congruence.

The foregoing results establish

Theorem 8. If a surface S_z, belonging to an inscribed sequence of the third order Σ_3, is an integral surface of the system of equations (6.17); then the line determined by points ξ and ζ, whose coordinates are defined by (6.32), generates a congruence of the sequence of Laplace in which is inscribed the one parameter family of sequences of Bompiani in which the sequence Σ_3 is inscribed.

As a matter of convenience in analysis, we have treated the above inscribed sequence of the third order in an unsymmetrical manner, in that we have represented the general surface S_z of the sequence by an unsymmetrical system of equations (6.17) and we have considered only the branch of the sequence which is initiated by the u-tangent lines of S_z. We shall now show that the v-tangent lines of S_z initiate a branch of the sequence Σ_3 which is symmetrical to the u-branch of that sequence. To this end, we shall establish

Theorem 9. If a surface S_z belongs to a sequence of order three, the generating point of S_z lies in the tangent plane of a conjugate net.

If S_z is transversal to the u-tangent lines of a surface S_x of a sequence of Bompiani, the coordinates x and z satisfy a relation

$$\mu z = x_u - \lambda x \qquad (\mu \neq 0).$$

Let S_η be the v-focal surface of the congruence of lines

conjugate to the parametric net of S_x. For a suitable choice of the coordinates η, they are related to the coordinates x by the following relation:

$$x = \eta_v + \phi\eta \qquad (\phi \neq \alpha),$$

where the coordinates η satisfy an equation of Laplace,

$$\eta_{uv} + \alpha\eta_u + \beta\eta_v + \gamma\eta = 0. \tag{6.34}$$

By eliminating x, from the above three equations, we obtain the relation

$$\mu z = (\phi - \alpha)\eta_u - (\lambda + \beta)\eta_v + (\phi_u - \gamma - \lambda\phi)\eta.$$

This equation shows that the point z is in the tangent plane of S_η. If $\lambda + \beta = 0$, the point z is on the u-tangent line of S_η. Then S_z is a surface of a sequence of Bompiani, and not of order three as by hypothesis. Hence the point z is distinct from the parametric tangent lines of S_η. The above proof holds when u and v are interchanged.

Theorem 10. The u-branch and the v-branch of an inscribed sequence of order three are inscribed symmetrically in two distinct one parameter families of sequences of Bompiani, and these sequences of Bompiani are inscribed in the same sequence of Laplace.

Let S_z be a surface of a sequence of order three, and let the generating point z be in the tangent plane of a conjugate net S_η, where the coordinates η satisfy the Laplace equation (6.34). The coordinates η are expressible as

$$z = k\eta_u + m\eta_v + n\eta. \qquad (km \neq 0).$$

A point y on the u-tangent line of S_η will have coordinates

(6.35) $y = \eta_u + \sigma'\eta$ $\qquad (\sigma' \neq \beta),$

and by equation (6.34)

$$y_v = -\alpha\eta_u + (\sigma' - \beta)\eta_v + (\sigma'_v - \gamma)\eta.$$

The surface S_y, being transversal to the congruence of u-tangent lines of S_η, is a surface of a sequence of Bompiani, or is itself a conjugate net. The v-tangent line of S_y goes through the point z if and only if the determinant of the coefficients in the right members of y, y_v, and z, has the value zero. By setting that determinant to zero, we have the following Riccati differential equation in σ':

(6.36) $m\sigma'_v = -k\sigma'^2 + (n + k\beta - q(m))\sigma' + m\gamma - n\beta.$

If σ' is a solution of (6.36), the point y does not generate a conjugate net when S_z belongs to an inscribed sequence of order three. If S_y were a conjugate net, S_z would be transversal to the lines of a congruence, and would belong to a sequence of Bompiani. Hence S_y belongs to a sequence of Bompiani which is inscribed in the sequence of Laplace to which S_η belongs. Each distinct solution σ' of equation (6.36) determines a distinct surface S_y which has the above properties and relations.

By interchanging the symbols u and v, and working with points on the v-tangent line of S_η, we prove by the above method that there exists a one parameter family of surfaces transversal to the v-tangent lines of S_η, and having u-tangent lines which pass through the point z. The u-branch of the sequence of surfaces S_z is inscribed in the sequences of

Bompiani which are initiated by the above surfaces.

Since the determination of ' and the coordinates y by equations (6.35) and (6.36) did not depend on the fact that S_z belongs to an inscribed sequence of the third order, we have established

Theorem 11. If a surface intersects each of the ∞^2 tangent planes of a conjugate net, that surface belongs to a sequence of order three, to a sequence of Bompiani, or to a sequence of Laplace.

7. Transformations which Generate Sequences of Order Three. In order to study the analytic form of the transformation which takes a surface S_z of a sequence of order three into an adjacent surface of the sequence, we note that the coefficient of z in the transformation (5.14), which is a transformation of this kind, is the same as the coefficient of z_{vvv} in each of equations (6.14), the first of (6.16), and the first of (6.17). By comparison, if a surface S_z is an integral surface of a system of differential equations (6.16), the transformations on S_z which initiate a sequence of the third order are

$$\begin{aligned} z_{-1} &= z_u - Az, \\ z_1 &= z_v - A''z. \end{aligned} \tag{7.10}$$

If at any point in a sequence of order three, the surfaces reduce to surfaces of a sequence of Bompiani, the conventional transformation for sequences of the third order becomes identical with the transformation of Bompiani. Let the

coordinates of surface S_{z_i} satisfy the system of equations

$$(7.11)\qquad z_{iuvv} = A_i z_{ivv} + (z_{iuv},\ z_{iu},\ z_{iv},\ z_i),$$
$$z_{iuuv} = A_i' z_{iuu} + (z_{iuv},\ z_{iu},\ z_{iv},\ z_i).$$

These coordinates also satisfy a system of equations (6.16), of which the first and third are

$$(7.12)\qquad z_{iuvvv} = A_i z_{ivvv} + (z_{iuvv},\ z_{ivv},\ z_{iuv},\ z_{iu},\ z_{iv},\ z_i),$$
$$z_{iuuuv} = A_i'' z_{iuuu} + (z_{iuuv},\ z_{iuu},\ z_{iuv},\ z_{iu},\ z_{iv},\ z_i).$$

By equations (4.8), (4.9), and (7.10), the transformations of Bompiani for equations (7.11) are the same as the conventional transformations in an inscribed sequence of order three, as determined for equations (7.12).

Theorem 12. Among the totality of sequences which can be inscribed in a given sequence of Bompiani, there is a subclass of sequences which are themselves sequences of Bompiani.

Let a surface S_x of a sequence of Bompiani be transversal to the v-tangent lines of the conjugate net S_η. Transversal to and conjugate to this same congruence of v-tangent lines is a second conjugate net S_ξ. The u-tangent lines of S_x and S_ξ both lie in the tangent plane of S_η. Hence, these lines intersect in a point y, which generates a surface S_y. Since S_y is transversal to the u-tangent lines of S_x, it is transformable into a sequence the u-branch of which is inscribed in the sequence of Bompiani initiated by S_x. But since S_y is also transversal to the congruence of u-tangent lines of the conjugate net S_ξ, S_y belongs to a sequence of Bompiani.

A further geometric characteristic of the transformations in the three kinds of sequences we have studied is described in the following.

Consider an inscribed sequence of order i(i = 1, 2, 3), where we understand sequences of orders one and two to be sequences of Laplace and Bompiani respectively. Let S_z be a surface belonging to a sequence of order i, and let S_ξ and S_ζ be any two surfaces transversal to the u-tangent lines of S_z. It will be shown that the u-tangent lines of S_ξ and S_ζ intersect in points, while the osculating spaces of i dimensions to the v-curves of S_ξ and S_ζ intersect in points. The transformation which sends S_z into $S_{z_{-1}}$ selects the special transversal surface S_ξ, the osculating S_i of these v-curves, intersect in points the osculating $S_{i_{-1}}$ of the v-curves of every other transversal surface S_ζ. We shall give a proof for the case i = 3, the cases i = 1, 2 being similar.

For i = 3, the coordinates z satisfy a system of equations (6.17), and from the geometric relations described above, the coordinates ξ and ζ can be expressed as

$$(7.13)\qquad \begin{aligned} \xi &= z_u - mz, \\ \zeta &= z_u - nz \end{aligned} \qquad (m \neq n).$$

The u-tangent lines of S_ξ and S_ζ are determined by the pairs of points having coordinates (ξ, ξ_u) and (ζ, ζ_u). These four functions are each expressible in terms of z, z_u, z_{uu}. Hence there is a linear relation of the form

$$(\xi, \xi_u, \zeta, \zeta_u) = 0.$$

This shows that the u-tangent lines of S_ξ and S_ζ intersect.

The osculating spaces of three dimensions to the respective v-curves of S_ξ and S_ζ are determined by the two sets of four points having coordinates $\xi, \xi_v, \xi_{vv}, \xi_{vvv}; \zeta, \zeta_v, \zeta_{vv}, \zeta_{vvv}$. By the derivatives of the left members of (7.13), these coordinates are expressed in terms of the derivatives of z as follows:

$$\xi = z_u - mz,$$
$$\xi_v = z_{uv} - mz_v - m_v z,$$
$$\xi_{vv} = z_{uvv} - mz_{vv} - 2m_v z_v - m_{vv} z,$$
$$\xi_{vvv} = z_{uvvv} - mz_{vvv} - 3m_v z_{vv} - 3m_{vv} z_v - m_{vvv} z,$$
$$\zeta = z_u - nz,$$
$$\zeta_v = z_{uv} - nz_v - n_v z,$$
$$\zeta_{vv} = z_{uvv} - nz_{vv} - 2n_v z_v - n_{vv} z,$$
$$\zeta_{vvv} = z_{uvvv} - nz_{vvv} - 3n_v z_{vv} - 3n_{vv} z_v - n_{vvv} z.$$

These eight functions are expressible, by means of (6.17), in terms of the seven functions z, z_u, z_v, z_{uv}, z_{vv}, z_{uvv}, z_{vvv}. Hence the eight functions on the left of the above equations satisfy a linear relation, which shows that the osculating spaces of three dimensions of the v-curves of S_ξ and S_ζ intersect in points, even when S_ξ and S_ζ are any two transversal surfaces of the u-tangent lines of S_z. If, however, we choose S_ξ specially by setting m = A, the four above derivatives of ξ and the first three derivatives of ζ are expressible in terms of the six functions z, z_u, z_v, z_{uv}, z_{vv}, z_{uvv}. Hence, there

is a linear relation

$$(\xi, \xi_v, \xi_{vv}, \xi_{vvv}, \zeta, \zeta_v, \zeta_{vv}) = 0.$$

This relation shows that the osculating spaces of three dimensions of the v-curves of S_ξ intersect the osculating planes of the corresponding v-curves of every other surface S_ζ which is transversal to the u-tangent lines of S_z. This particular surface S_ξ is the minus first transform of S_z. Similar facts hold for u and v interchanged.

8. Determination of the Integral Surfaces. If after any number of successive applications of the transformations (7.10) to equations (6.16) or (6.17) one arrives at a system of equations the integral surfaces of which degenerate to points or curves, the integral surfaces of the original equations can be determined by successive quadratures, as by the classical method of Laplace. Moreover, it is possible to make direct use of the method of Laplace in seeking these integral surfaces, as indicated by

Theorem 13. The problem of determining the integral surfaces of a system of differential equations (6.17), or the equivalent system (6.16), is reducible to the problem of solving a single differential equation of Laplace, and a single Riccati differential equation.

By means of equations (6.32), and (6.33), the coordinates η of the generating point of a focal surface S_η of the sequence of Laplace associated with an integral surface S_z,

are expressed in terms of z and the coefficients of equations (6.17). These coordinates η satisfy a differential equation of Laplace (6.34), the coefficients of which can be determined by eliminating z from the derivatives of η by (6.17).

From the solutions η of the Laplace differential equation (6.34), the coordinates ξ and ζ are defined by equations (6.33). The coordinates x are defined in terms of ξ, ζ, and a solution γ of the Riccati differential equation (6.28), by means of (6.31) and (6.32). Finally, we obtain the solutions z of (6.17), expressed in terms of the coordinates x by means of (6.23), in which λ and μ are defined by equations (6.29).

CHAPTER IV

PARABOLIC AND IDENTICAL CORRESPONDENCE

9. Parabolic Correspondence. In Article 2 it was shown that for a proper choice of the parametric system, a parabolic correspondence between two surfaces S_x and S_y can be represented by a system of differential equations (2.4), which we rewrite here for convenience as

$$(9.1)\qquad \begin{aligned} y_u - Ay &= fx + mx_u + Sx_v \qquad (S \neq 0), \\ y_v - By &= gx + mx_v. \end{aligned}$$

By taking the v and u derivatives respectively of the first and second of equations (9.1), it can be shown that if the v-curves of S_x are not an asymptotic family, the functions y, y_u, y_v, and y_{uv} are expressible in terms of the four functions x, x_u, x_v, x_{vv}. By imposing the integrability conditions $\partial y/\partial u = y_u$, and $\partial y/\partial v = y_v$ on these expressions, it is determined that the coordinates x satisfy an integrable system of third order partial differential equations of the forms

$$(9.2)\qquad \begin{aligned} x_{uvv} &= ax_{uu} + 2bx_{uv} + cx_{vv} + dx_u + ex_v + lx, \\ x_{vvv} &= a'x_{uu} + 2b'x_{uv} + c'x_{vv} + d'x_u + e'x_v + l'x. \end{aligned}$$

Equations (9.2) require the following conditions of integrability:

$$
\begin{aligned}
&a' = 2b'-a = 0,\\
&ac'-a_v+d'-2ab = 0,\\
&2b'_u+2bc'-2b'c+e-2b_v-4b^2-d = 0,\\
&c'_u-c_v-e-2bc = 0,\\
&d'_u+c'd-cd'-2bd-d_v+l' = 0,\\
&e'_u+c'e-ce'-2be-e_v-l = 0,\\
&l'_u+c'l-cl'-2bl-l_v = 0.
\end{aligned}
\tag{9.3}
$$

Theorem 14. If a surface S_x is in parabolic correspondence with a second surface, the coordinates x satisfy an integrable system of differential equations (9.2); and conversely.

To prove the converse of this theorem we express a system of coordinates y in terms of undetermined coefficients as

$$y = x_{vv}-px_u-qx_v+rx.$$

If the coordinates x do not satisfy a differential equation of the second order, the above coordinates y satisfy equations (9.1) when and only when $p = a$, $q = 2b$, and

$$y = x_{vv}-ax_u-2bx_v+rx. \tag{9.4}$$

Hence the generating points of all surfaces in parabolic correspondence with S_x lie on a straight line. Of the totality of points y represented in (9.4), only one generates a focal surface. The coordinates η of the focal point are determined by a special value of r in (9.4) to be

$$\eta = x_{vv}-ax_u-2bx_v+(a_u-ac-d)x. \tag{9.5}$$

By means of equations (9.2) it can be shown that the coordinates γ and x satisfy the following relations:

$$\gamma_u - c\gamma = fx + gx_v,$$
$$\gamma_v - (c' - b)\gamma = gx. \tag{9.6}$$

in which

$$f = 1 + \partial(a_u - ac - d)\partial_u - c(a_u - ac - d),$$
$$g = 1' + \partial(a_u - ac - d)\partial_v - (b - c')(a_u - ac - d),$$
$$s = -b_u - 2bc + e.$$

Equations (9.6) show that the surface S_γ is a focal surface of the ∞^2 lines $x\gamma$, unless $g = 0$. In that case S_γ degenerates to a point or a curve. When $g \neq 0$, the developable surfaces formed by the v-tangent lines of S_γ cut out the v-curves on S_x.

By using the second of equations (9.6) to eliminate x from the first of that system, it is shown that the coordinates γ satisfy a differential equation of the form

$$\gamma_{uv} = \alpha\gamma_u + \beta\gamma_v + \gamma. \tag{9.8}$$

This equation indicates that the v-curves of S_γ are an asymptotic family. Hence, we have

Theorem 15. *Corresponding points of two surfaces in a parabolic correspondence are joined by the tangent lines of one family of asymptotic curves of a surface.*

Theorem 16. *If two surfaces are transversal to the ∞^2 tangent lines of a family of asymptotic curves of a surface, the two surfaces are in parabolic correspondence.*

Let S_γ be the surface bearing the family of asymptotic curves, and let the coordinates γ satisfy equation (9.6). If

S_x and S_y are transversal to the v-tangent lines of S_η, the coordinates x and y are expressible as

$$x = \eta_v + \lambda\eta,$$
$$y = \eta_v + \mu\eta \qquad (\lambda \neq \mu).$$

By computing the derivatives of x and y, and reducing the derivatives of η by means of equation (9.6), it can be shown that the coordinates x and y satisfy a system of equations (9.1), which indicate that S_x and S_y are in parabolic correspondence.

10. Identical Correspondence. In Article 2, it was shown that for a suitable choice of the parametric curves, an identical correspondence between two surfaces S_x and S_y can be represented by a system of differential equations

$$\begin{aligned} y_u - Ay &= fx + mx_u, \\ y_v - By &= gx + mx_v. \end{aligned} \tag{10.1}$$

By taking the u derivative of the second and the v derivative of the first of (10.1), we obtain two new equations which together with (10.1) form four nonhomogeneous linear equations in y, y_u, y_v, y_{uv}. By solving these equations formally, we obtain, on setting $A_v - B_u = \Delta$,

$$\begin{aligned} \Delta y &= (g_u - f_v + Bf - Ag)x + (g - m_v + Bm)x_u + (m_u - f - Am)x_v, \\ \Delta y_u &= (f(A_v - B_u + AB) + A(g_u - f_v - Ag))x + (m(A_v - B_u + AB) + A(g - m_v))x_u \\ &\quad + A(m_u - f - Am)x_v, \\ \Delta y_v &= (g(A_v - B_u - AB) + B(Bf + g_u - f_v))x + B(Bm + g - m_v)x_u \\ &\quad + (m(A_v - B_u - AB) + B(m_u - f))x_v. \end{aligned} \tag{10.2}$$

If equations (10.2) do not vanish identically, the two surfaces S_x and S_y have the same tangent planes, and the lines of intersection of corresponding tangent planes are indeterminate. By imposing integrability conditions on equations (10.2), we obtain the system of differential equations

$$(10.3)\qquad \begin{aligned} (g-m_v+Bm)x_{uu}+(m_u-f-Am)x_{uv}+(x_u, x_v, x) &= 0, \\ (m_u-f-Am)x_{vv}+(g-m_v+Bm)x_{uv}+(x_u, x_v, x) &= 0. \end{aligned}$$

These equations show that S_x is either developable or a surface in a space of three dimensions.

If equations (10.2) vanish identically, we have

$$(10.4)\qquad \begin{aligned} A_v-B_u &= g-m_v+Bm = 0, \\ m_u-f-Am &= g_u-f_v-Ag+Bf = 0. \end{aligned}$$

Consider the point ξ on the line xy, which has coordinates

$$\xi = y-mx.$$

If we use this equation to eliminate y from equations (10.1), we obtain the relations

$$\begin{aligned} \xi_u-A\xi &= (f-m_u+Am)x, \\ \xi_v-B\xi &= (g-m_v+Bm)x. \end{aligned}$$

By (10.4) the right members of these equations vanish, showing that the point ξ is stationary. These results establish

Theorem 17. If two non-developable surfaces in a space of n dimensions are in an identical correspondence, corresponding points of the two surfaces are connected by the lines of a bundle.

The converse of Theorem 17 is also true. To show this,

let S_x and S_y have corresponding points joined by lines of a bundle the base of which is the point having coordinates α. The coordinates x and y satisfy a relation

$$\mu x = \lambda y + \alpha.$$

By differentiation,

$$\mu x_u + \mu_u x = \lambda_u y + \lambda y_u,$$

$$\mu x_v + \mu_v x = \lambda_v y + \lambda y_v.$$

This is a system of equations of the form of (10.1), which represents an identical correspondence between S_x and S_y.

If the points of an arbitrary surface be joined by straight lines to a stationary point, that surface is in identical correspondence with every surface which is transversal to the bundle of lines thus formed.

VITA

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During his residence at the University of Chicago, he has had courses with Professors Moore, Bliss, Lane, Logsdon, Macmillan, Slaught, Dickson, Bartky, Barnard, and others. To each of these, he wishes to acknowledge his indebtedness, and to express his sincere appreciation. He is especially grateful to Professor Lane, who suggested the problem of this thesis, and directed its development.